μ - Games

Mathematics Utrecht

Utrecht University

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Exercise 1: Fermat Theory

- We want to find $f(x+y) = g(x) + yh(x,y)$.
- Just work it out!

$$f(x+y) = \sum_{k=0}^{n} a_k (x+y)^k$$

- Be careful with indices and find:

$$g(x) = f(x),\ h(x,y) = \sum_{k=1}^{n} \sum_{l=0}^{k-1} C_l^k a_k x^l y^{(k-1)-l}$$

- Evaluating h at $y = 0$, we find:

$$h(x,0) = \sum_{k=1}^{n} k a_k x^{k-1} = f'(x)$$

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Exercise 2: Hilbert's Hotel Lite

Is $n + 1$ prime?
Given: number of ways to order n guests over n rooms, module $n + 1$.
That is $n! \mod n + 1$
If $n + 1$ not prime, all its factors are in $n!$
If $n! \mod n + 1 \equiv 0$, we know $n + 1$ not prime. For all non-primes except $n + 1 = 4$.

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Exercise 3: Limit Maps

- $\lim_{n \to \infty} f_n = x$.
- Arnold's Cat Map (Vladimir Arnold).
- Periodic for $q \in \mathbb{Q}$.
- Brute-force is not good enough.
- Calculate period by keeping track of the numerator and denumerator.

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Exercise 4: Zeckendorf Cover

- How many different sets of numbers can be made from first n Fibonacci numbers using Zeckendorf decomposition?
- Recognize exact numbers do not matter.
- Need to count number of ways to partition $[n] = \{1, ..., n\}$ such that no sets in a partition contain consecutive numbers.
- i.e. for $n = 3$, two options:
  - {1}, {2}, {3}
  - {1, 3}, {2}
- Cleverly count these partitions!

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- Let Z_n denote the number of valid partitions of $[n]$. We want to know $|Z_n|$.
- Partitions at Z_{n+1} arise in two ways:
- Either $\{n+1\}$ is added to a partition $S \in Z_n \implies 1$ option
- Or $n+1$ is added to a $A \in S \implies |S|$ options?
- No! $|S|-1$ options
- Then, combining this, we get

$$|Z_{n+1}| = \sum_{S \in Z_n} (1 + |S| - 1) = \sum_{S \in Z_n} |S|$$

- Finding all partitions $O(n!)$, too slow!

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- Instead, define $N_n[k]$ number of valid partitions of $[n]$ of size k.
- Then, $|Z_{n+1}| = \sum_{k=1}^{n} kN_n[k]$
- Calculate table $N_n[k]$
- Use two options again:
- Either $\{n+1\}$ is added to a partition $S \in Z_n \implies 1$ option
- Or $n+1$ is added to a $A \in S \implies |S| - 1$ options
- So, $N_n[k] = N_{n-1}[k-1] + (k-1)N_{n-1}[k]$
- **Caution:** Do modulo $10^9 + 7$ a lot and watch out for $n = 1$.

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Exercise 5: Graph Magic

First, let us look at the case that the eigenvalues are symmetric.
Then if $\lambda_1, ..., \lambda_n$ be our eigenvalues. Then

$$\sum_{i=1}^{n} \lambda_n = 0.$$

For k odd integer:

$$\sum_{i=1}^{n} \lambda_n^k = 0.$$

We know $Tr(A) = \sum_{i=1}^{n} \lambda_n = 0$ and also $Tr(A^k) = \sum_{i=1}^{n} \lambda_n^k = 0$.
We can interpret this as walks from an vertice to itself of length k. Thus A must be bipartite.

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Other way around: Bipartite $\Rightarrow$ symmetric?

If A bipartite, we have: $A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$.

Now if $(x, y)^T$ is an eigenvector with eigenvalue λ:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

so $B^T x = \lambda y$ and $By = \lambda x$. We see:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -By \\ B^T x \end{pmatrix} = -\lambda \begin{pmatrix} x \\ -y \end{pmatrix}.$$

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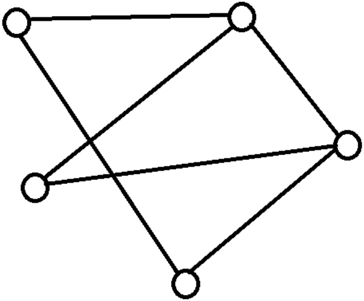

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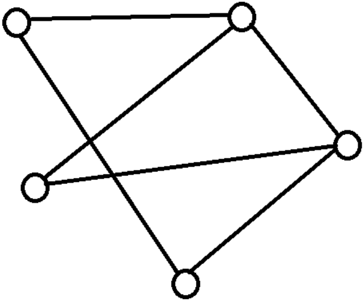

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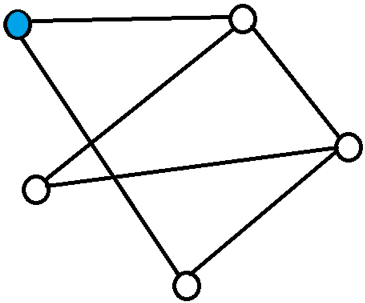

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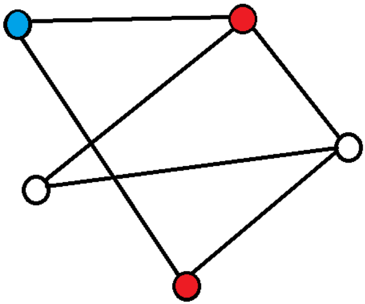

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